

## Strong Comparison Principle for Radial Solutions of Quasi-Linear Equations

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Let  $\Omega$  be either a ball or an annulus centered about the origin in  $\mathbb{R}^N$  and  $\Delta_p$  the usual  $p$ -Laplace operator in  $\mathbb{R}^N$ . Let  $f_1, f_2 \in L^1_{loc}(\Omega)$  be two radial functions on  $\Omega$  with  $f_1 \leq f_2, f_1 \not\equiv f_2$ . Let  $b : \mathbb{R} \rightarrow \mathbb{R}$  be a non-decreasing continuous function. Let  $u_1, u_2 \in C^{1,\beta}(\Omega)$ ,  $\beta \in (0, 1]$  be any two radial weak solutions of  $-\Delta_p u_i = b(u_i) + f_i$  in  $\Omega$ . We then show that  $u_1 \leq u_2$  in  $\Omega$  implies  $u_1 < u_2$  in  $\Omega$  and also that appropriate versions of Hopf boundary point principle hold. © 2001 Academic Press

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### 0. INTRODUCTION

Let  $p \in (1, \infty)$  and let  $-\Delta_p$  denote the usual  $p$ -Laplace operator in  $\mathbb{R}^N$  defined by  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ . Let  $\Omega \subset \mathbb{R}^N, N \geq 1$ , be a bounded connected domain whose boundary  $\partial\Omega$  is a  $C^2$ -manifold if  $N \geq 2$ . Let  $b : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function which is monotone in the second variable. We are interested in the following:

**Strong Comparison Principle (SCP).** Take  $f_1, f_2 \in L^1_{loc}(\Omega), f_1 \leq f_2$ , and  $f_1 \not\equiv f_2$ . Let  $u_1, u_2 \in C^{1,\beta}(\Omega)$ ,  $\beta \in (0, 1]$ , be any two weak solutions of

$$(P) \quad -\Delta_p u_i = b(x, u_i) + f_i(x) \text{ in } \Omega, i = 1, 2.$$

We say that the strong comparison principle (SCP) holds for the problem (P) if

$$u_2 \geq u_1 \quad \text{in } \Omega \quad \Rightarrow \quad u_2 > u_1 \quad \text{in } \Omega.$$



We now give a quick review of known results concerning (SCP) for the problem (P). We refer to [1] for more details and references to related works in the area. In this interesting reference, the following results are shown (see Theorem 2.1 in [1]):

(i) Let  $b(x, u)$  be nondecreasing in the second variable. Suppose the following problem admits a unique nonnegative solution for any  $f \in L^\infty(\Omega)$ ,  $f \geq 0$  in  $\Omega$ :

$$\begin{aligned} -\Delta_p u &= b(x, u) + f(x) \text{ in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Then SCP holds if  $\partial\Omega$  is a connected manifold when  $N \geq 2$  and  $\Omega$  is an interval when  $N = 1$  with additional assumptions:  $f_2 \geq f_1 \geq 0$ ,  $u_1 = u_2 = 0$  on  $\partial\Omega$ ,  $u_1 \geq 0$ , and  $u_2 \geq 0$  in  $\Omega$ .

(ii) Let  $b(x, u)$  be a locally Lipschitz function, nonincreasing in the second variable, with the following behaviour when  $|u| \rightarrow 0$ :

$$\limsup_{|u| \rightarrow 0} \sup_{x \in \bar{\Omega}} \left| \frac{\partial b}{\partial u}(x, u) \right| \leq c \begin{cases} |u|^{p-2} & \text{if } 1 < p < 2 \\ 1 & \text{if } p \geq 2 \end{cases}$$

for some  $c > 0$ . Then SCP holds in any dimension  $N \geq 1$  provided that  $1 < p \leq 2$ ,  $\Omega$  is a ball (or an interval in  $\mathbb{R}$ ) in  $\mathbb{R}^N$  centered about the origin,  $b(x, u)$ ,  $f_1(x)$ ,  $f_2(x)$ ,  $u_1(x)$ , and  $u_2(x)$  are all radial functions of the variable  $x$  on  $\Omega$ , and  $u_1$ , and  $u_2$  are nonnegative on  $\Omega$  with zero boundary values. Also, if  $p > 2$  and  $\frac{\partial b}{\partial u}$  is a large enough negative number, then a counterexample to SCP is shown.

In this paper we concentrate only on the case when  $b(x, u)$  is a non-decreasing function in the second variable and radial in the first variable. We show that by restricting ourselves to radial solutions on a ball or an annulus, SCP holds without any additional assumptions.

## 1. THE MAIN LEMMA

Fix  $0 \leq r < R \leq \infty$ ,  $c \geq 0$ . Let  $\alpha, \beta : [r, R] \times \mathbb{R} \rightarrow \mathbb{R}$  be continuous functions, with  $\alpha$  strictly increasing and  $\beta$  nondecreasing in the second variable. We then have the following.

LEMMA 1. Let  $f_1, f_2 \in L^1_{loc}(r, R)$ ,  $f_2 \geq f_1$ , and  $f_1 \not\equiv f_2$ . Let  $u_1, u_2 \in C^{1,\beta}[r, R]$  solve, in the weak sense,

$$\begin{cases} -(\alpha(x, u'_i))' - (\frac{c}{x})\alpha(x, u'_i) = \beta(x, u_i) + f_i(x), & x \in (r, R), i = 1, 2 \\ u'_i(0) = 0, & i = 1, 2 \text{ if } c > 0 \text{ and } r = 0. \end{cases}$$

Then SCP holds; that is,

$$u_2 \geq u_1 \text{ in } (r, R) \Rightarrow u_2 > u_1 \text{ in } (r, R).$$

Furthermore, the following boundary Hopf lemma holds: When  $0 < r < R \leq \infty$  and  $c \geq 0$ ,  $u_1(r) = u_2(r) \Rightarrow u'_1(r) < u'_2(r)$ , and if  $R < \infty$ , then  $u_1(R) = u_2(R) \Rightarrow u'_1(R) > u'_2(R)$ . When  $r = 0$  and  $c > 0$ , and if  $R < \infty$ , then  $u_1(R) = u_2(R) \Rightarrow u'_1(R) > u'_2(R)$ .

When  $r = 0$  and  $c = 0$ ,  $u_1(0) = u_2(0) \Rightarrow u'_1(0) < u'_2(0)$  and if  $R < \infty$ , then  $u_1(R) = u_2(R) \Rightarrow u'_1(R) > u'_2(R)$ .

*Proof.* By definition,  $\forall \phi \in C_0^1(r, R)$ , we have

$$(I_i) \quad \int_r^R \alpha(x, u'_i) \phi' = c \int_r^R \frac{\alpha(x, u'_i)}{x} \phi + \int_r^R \beta(x, u_i) \phi + \int_r^R f_i \phi, \quad i = 1, 2.$$

Now  $(I_2) - (I_1)$  gives

$$\begin{aligned} \int_r^R [\alpha(x, u'_2) - \alpha(x, u'_1)] \phi' &= c \int_r^R \frac{[\alpha(x, u'_2) - \alpha(x, u'_1)]}{x} \phi \\ &\quad + \int_r^R [\beta(x, u_2) - \beta(x, u_1)] \phi + \int_r^R (f_2 - f_1) \phi. \end{aligned}$$

Define the functions  $G, f^*: [r, R] \rightarrow \mathbb{R}$  by  $G(x) = \alpha(x, u'_2(x)) - \alpha(x, u'_1(x))$ ,  $f^*(x) = \beta(x, u_2(x)) + f_2(x) - \beta(x, u_1(x)) - f_1(x)$ . Note that the hypotheses  $u_2 \geq u_1$ ,  $\beta(x, \cdot)$  is a nondecreasing function, and  $f_2 \geq f_1$ , together imply  $f^* \geq 0$ . We can now rewrite the above equation as

$$(*) \quad \int_r^R G \phi' = c \int_r^R \frac{G}{x} \phi + \int_r^R f^* \phi, \quad \phi \in C_0^1(r, R).$$

*Claim.*  $(*)$  implies that the function  $x^c G(x)$  is nonincreasing in  $[r, R]$ .

*Proof.* Let  $r < x_0 < y_0 < R$ . Define for  $\varepsilon > 0$ , a piecewise smooth  $\phi_\varepsilon$  in  $(r, R)$  by

$$\phi_\varepsilon(x) = \begin{cases} 0, & x \in (r, x_0 - \varepsilon) \cup (y_0 + \varepsilon, R) \\ \varepsilon, & x \in [x_0, y_0] \\ \text{"linear"} & x \in [x_0 - \varepsilon, x_0) \cup (y_0, y_0 + \varepsilon]. \end{cases}$$

Plugging in such a  $\phi_\varepsilon$  into  $(*)$ , we get

$$\int_{x_0 - \varepsilon}^{x_0} G - \int_{y_0}^{y_0 + \varepsilon} G = c \int_{x_0 - \varepsilon}^{y_0 + \varepsilon} \frac{G}{x} \phi_\varepsilon + \int_{x_0 - \varepsilon}^{y_0 + \varepsilon} f^* \phi_\varepsilon.$$

Dividing by  $\varepsilon$  on both sides and letting  $\varepsilon \rightarrow 0$ , we get

$$(**) \quad G(x_0) - G(y_0) = c \int_{x_0}^{y_0} \frac{G}{x} + \int_{x_0}^{y_0} f^*.$$

Now fixing  $y_0 \in (r, R)$  and thinking of  $x_0$  as varying in any compact interval  $I \subset (r, R)$ , we obtain from (\*\*) that  $G$  is absolutely continuous on  $I$ . Hence  $G$  is differentiable a.e. on  $(r, R)$ . Choose  $h > 0$  and  $x$  any point of differentiability for  $G$ . Set  $x_0 = x$ ,  $y_0 = x + h$  in (\*\*), divide by  $h$ , and pass to the limit  $h \rightarrow 0$  to obtain

$$-G'(x) = \left(\frac{c}{x}\right)G(x) + f^*(x) \quad \text{a.e. } x \in (r, R).$$

Multiplying by the integrating factor  $x^c$  on either side, for a.e.  $x \in (r, R)$  we obtain

$$[x^c G(x)]' = -x^c f^*(x) \leq 0.$$

The claim now follows. ■

Define  $w(x) = (u_2 - u_1)(x)$ . Then, by hypotheses,  $w \geq 0$  in  $(r, R)$ . We wish to show  $w > 0$  in  $(r, R)$ . Suppose  $w(x_0) = 0$  for some  $x_0 \in (r, R)$ . We show that this leads to a contradiction. Clearly,  $x_0$  is a point of global minimum for  $w$  in  $(r, R)$ , and hence  $w'(x_0) = 0$ . Hence at  $x_0$ :  $u_1(x_0) = u_2(x_0)$ ,  $u_1'(x_0) = u_2'(x_0)$ . This means  $G(x_0) = 0$ . From the claim, since  $x^c G(x)$  is nonincreasing on  $(r, R)$ , it follows that  $G(x) \geq 0$  in  $(r, x_0)$  and  $G(x) \leq 0$  in  $(x_0, R)$ . Since  $\alpha$  is a strictly increasing function in the second variable, it follows  $u_2' \geq u_1'$  in  $(r, x_0)$ , and  $u_2' \leq u_1'$  in  $(x_0, R)$ . Thus  $w' \geq 0$  in  $(r, x_0)$  and  $w' \leq 0$  in  $(x_0, R)$ . This means that  $w$  is a nonnegative function on  $(r, R)$  which vanishes at  $x_0$  and which is nondecreasing to the left of  $x_0$  and nonincreasing to the right of  $x_0$ . This forces  $w \equiv 0$  in  $(r, R)$ , which implies  $u_1 \equiv u_2$  and hence  $f_1 \equiv f_2$  in  $(r, R)$ , which is a contradiction. This contradiction shows that  $u_2 > u_1$  in  $(r, R)$ .

We now show the Hopf boundary point principle. We suppose that  $u_1(r) = u_2(r)$  and  $u_1'(r) = u_2'(r)$ , and show that this leads to a contradiction. We now have  $G(r) = 0$ , and hence  $G \leq 0$  in  $(r, R)$  by the claim. This means  $w(r) = 0$  and  $w$  is nonincreasing to the right of  $r$  and hence  $w \equiv 0$  in  $(r, R)$ . As before, we obtain a contradiction to the assumption  $f_1 \not\equiv f_2$ . Hence  $u_1(r) = u_2(r) \Rightarrow u_1'(r) < u_2'(r)$ . Similarly, if  $R < \infty$ ,  $r \geq 0$ , and  $c \geq 0$ , then we obtain  $u_1(R) = u_2(R)$  and  $u_1'(R) = u_2'(R) \Rightarrow G(R) = 0$  and  $G \geq 0$  in  $(r, R)$ . Hence as before,  $w \equiv 0$  in  $(r, R)$ , which again leads to the same contradiction. Therefore, if  $R < \infty$ , then  $u_1(R) = u_2(R) \Rightarrow u_1'(R) > u_2'(R)$ . This proves the lemma. ■

## 2. STRONG COMPARISON PRINCIPLE FOR QUASI-LINEAR EQUATIONS IN A BALL OR AN ANNULUS

In this section we assume that the domain  $\Omega$  is either a ball centered at the origin or an annulus symmetric about the origin in  $\mathbb{R}^N$ . We also allow

the limiting cases when  $\Omega = \mathbb{R}^N$  (a ball of infinite radius) and the exterior of a ball in  $\mathbb{R}^N$  (an annulus with an infinite outer radius). Accordingly, we may take  $\Omega = B(0, R)$  or  $\Omega = B(0, R) \setminus \overline{B(0, r)}$  for  $0 \leq r < R \leq \infty$ . Let  $J : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  be a continuous function with the following structure:

$$J(x, \nabla u(x)) = \alpha(|x|, u'(|x|)) \frac{x}{|x|}, \quad \forall u \in C^1(\Omega), \quad u \text{ radial},$$

where  $\alpha : [r, R] \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function that is strictly increasing in the second variable. Let  $b : [r, R] \times \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function that is nondecreasing in the second variable. Then, as a direct application of Lemma 1, we have the following SCP results.

**THEOREM 1.** *Let  $\Omega = B(0, R)$ ,  $0 < R \leq \infty$ . Let  $f_1, f_2 \in L^1_{loc}(\Omega)$  be two radial functions on  $\Omega$  with  $f_2 \geq f_1$ , and  $f_1 \not\equiv f_2$  in  $\Omega$ . Let  $u_1$ , and  $u_2$  be any two  $C^{1,\beta}(\Omega)$ ,  $\beta \in (0, 1]$ , radial weak solutions of*

$$-\operatorname{div}(J(x, \nabla u_i(x))) = b(|x|, u_i) + f_i(x) \text{ in } \Omega, \quad i = 1, 2.$$

*If  $u_2 \geq u_1$  in  $\Omega$ , then  $u_2 > u_1$  in  $\Omega$ . Furthermore, the following Hopf lemma holds: If  $R < \infty$  and  $u_1(R) = u_2(R)$ , then  $u'_1(R) > u'_2(R)$ .*

*Proof.* A short computation gives, for  $0 < |x| < R$ ,  $-\operatorname{div}(J(x, \nabla u_i(x))) = -(\alpha(|x|, u'_i(|x|)))' - \frac{(N-1)}{|x|} \alpha(|x|, u'_i(|x|))$ . Also, since  $u_i$  are  $C^1$  radial functions,  $u'_1(0) = u'_2(0) = 0$ . Hence we may appeal to Lemma 1 with  $c = N - 1$  to conclude the theorem.

**THEOREM 2.** *Let  $\Omega = B(0, R) \setminus \overline{B(0, r)}$ ,  $0 < r < R \leq \infty$ . Let  $f_1, f_2, u_1$ , and  $u_2$  be as in the statement of Theorem 1. If  $u_2 \geq u_1$  in  $\Omega$ , then  $u_2 > u_1$  in  $\Omega$ . Furthermore, the following Hopf lemma holds:  $u_1(r) = u_2(r) \Rightarrow u'_1(r) < u'_2(r)$ ; if  $R < \infty$ ,  $u_1(R) = u_2(R) \Rightarrow u'_1(R) > u'_2(R)$ .*

*Proof.* Follows similarly by appealing to Lemma 1.

## REFERENCE

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